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THE NONAXISYMMETRIC CONTACT THERMOELASTIC PROBLEM FOR A HALF-SPACE WITH A MOTIONLESS RIGID SPHERICAL INCLUSION

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Based on the generalized Fourier method used simultaneously in the spherical and cylindrical systems of coordinates, we suggested an analytical method for solving the contact problem of thermoelasticity for an elastic half-space with a rigid spherical inclusion. The problem is reduced to an infinite system of linear algebraic equations with the Fredholm operator provided that the boundary surfaces do not intersect. An approximate solution of the system in the form of a series with respect to a small parameter is obtained. The numerical analysis of the problem is presented.

For the first time, the technique of the generalized Fourier method (GFM) for solving thermoelasticity problems in the case of multiply connected bodies was used in the solution of problems of thermoelasticity for a half-space with a spherical cavity (the axisymmetric formulation) in [1]. Then this technique was developed in [2] in solution of the thermoelasticity problem for a sphere with a spherical cavity. In the present work, the GFM has been developed for the general nonaxisymmetric case.

We consider the static thermoelastic contact problem of nonaxisymmetric frictionless indentation, of an in-plan round die of radius a into an elastic half-space with the heated absolutely rigid motionless spherical inclusion the center of which coincides with the axis of the die and is at a distance h from the boundary of the half-space. The temperature distribution in the half-space is assumed to be stationary, at infinity and on a plane surface the temperature is $T = 0$, and the surface of the inclusion has a constant temperature $T = T_0$.

We introduce the cylindrical (ρ, z, φ) and spherical (r, θ, φ) systems of coordinates with a common axis of symmetry, which coincide with the centers of the die basis and the inclusion. It is assumed that the projection of the point of application of force P, which affects the die, onto the boundary of the half-space has the coordinates $(\rho_0, 0, 0)$. The temperature distribution and the stressed-deformed state (SDS) are determined from the solution of the nonbound stationary problem of thermoelasticity

$$\nabla^2 T = 0\,, \quad T(\rho, 0, \varphi) = 0\,, \quad T(R, \theta, \varphi) = T_0\,; \tag{1}$$

$$\nabla^2 \mathbf{U} + (1 - 2\sigma)^{-1}\, \nabla\,(\nabla \mathbf{U}) = \frac{2 + 2\sigma}{1 - 2\sigma}\, \alpha_T \nabla T\,; \tag{2}$$

$$\tau_{\rho z}(\rho, 0, \varphi) = \tau_{\varphi z}(\rho, 0, \varphi) = 0\,,$$

$$U_z(\rho, 0, \varphi) = \delta + \gamma\rho \cos\varphi\,, \quad \rho < a\,, \quad \sigma_z(\rho, 0, \varphi) = 0\,, \quad \rho > a\,; \tag{3}$$

$$\mathbf{U}(R, \theta, \varphi) = 0\,. \tag{4}$$

In view of the axial symmetry of the heat-conduction problem (1), we look for its solution in the region $\Omega = \{z > 0, r > R\}$ in the form of the superposition of the outer basis solutions of the Laplace equation for a sphere and the inner solutions for a half-space:

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$$T = \int_0^\infty B(\lambda)\, u_{\lambda,0}^{-(2)}\, d\lambda + \sum_{n=0}^\infty b_n \frac{R^{n+1}}{n!} u_{n,0}^{+(4)}\,, \tag{5}$$

where

$$u_{\lambda,m}^{\pm(2)} = \exp(\pm \lambda z + im\varphi)\, J_m(\lambda\rho)\,;$$

$$u_{n,m}^{\pm(4)} = \begin{Bmatrix} \dfrac{(n-m)\,!}{r^{n+1}} \\ \dfrac{r^n}{(n+m)\,!} \end{Bmatrix} P_n^m(\cos\theta)\exp(im\varphi)\,.$$

Use of the GFM (see [3]) reduces the stationary heat-conduction problem to a system of equations relative to $B(\lambda)$, b_n:

$$B(\lambda) + \exp(-\lambda h) \sum_{n=0}^\infty b_n \frac{(-1)^n}{n!} R^{n+1} \lambda^n = 0\,,$$

$$b_k + \sum_{n=0}^\infty b_n (-1)^{n+k+1} \frac{(n+k)\,!}{n!k!} \left(\frac{R}{2h}\right)^{n+k+1} = T_0 \delta_{k0}\,, \quad k = 0, 1, 2, \dots\,.$$

Then we consider the boundary-value problem (2)–(4) in the region $\boldsymbol{\Omega}$. It is known that a general solution of the inhomogeneous Lamé equation (2) can be presented in the form of the sum of the general solution of the homogeneous equation ($\mathbf{U}_0$) and the partial solution of the inhomogeneous Lamé equation ($\mathbf{U}_1$). We look for $\mathbf{U}_1$ in the form $\mathbf{U}_1 = \nabla\Phi$. In this case, it is not difficult to show that the thermoelastic potential Φ satisfies the Poisson equation

$$\nabla^2 \Phi = \frac{1+\sigma}{1-\sigma} \alpha_T T\,.$$

An explicit form of the function T (5) allows one to recover the potential Φ; then the partial solution has the form

$$\mathbf{U}_1 = -\sigma_\alpha \left\{ \int_0^\infty B(\lambda)\, \nabla \left[\frac{\lambda z + 1}{\lambda^2} u_{\lambda,0}^{-(2)} \right] d\lambda + \sum_{n=0}^\infty b_n \frac{R^{n+1}}{2n-1} \nabla \left[r^2 u_{n,0}^{+(4)} \right] \right\},$$

$$\sigma_\alpha = \frac{1+\sigma}{2(1-\sigma)} \alpha_T\,.$$

For presentation of $\mathbf{U}_1$ in cylindrical and spherical coordinates we use the following lemmas.

Lemma 1. When $n \ge 1$, $z_1 \ne 0$, the integral presentation holds:

$$r_1^2 u_{n,0}^{+(4)} = \frac{(\pm 1)^n}{n!} \int_0^\infty u_{\lambda,0}^{\mp(2)} \left[\pm (2n-1)\lambda z_1 - (n-1)^2 \right] \lambda^{n-2}\, d\lambda\,.$$

Lemma 2. When $\lambda \ne 0$, the expansion holds:

$$\frac{\lambda z + 1}{\lambda^2} u_{\lambda,0}^{-(2)} = \exp(-\lambda h) \sum_{k=0}^\infty (-1)^k \lambda^k \left[\frac{h}{\lambda} - \frac{k^2 - 1}{(2k-1)\lambda^2} - \frac{r^2}{2k+3} \right] u_{k,0}^{-(4)}\,.$$

The results of the lemmas make it possible to write $\mathbf{U}_1$ in the cylindrical and spherical systems of coordinates:

$$\mathbf{U}_1 = -\sigma_\alpha \int_0^\infty \Big\{ B(\lambda) \exp(-\lambda z) \Big[-\frac{\lambda z + 1}{\lambda} J_1(\lambda\rho)\, \mathbf{e}_\rho - z J_0(\lambda\rho)\, \mathbf{e}_z \Big] -$$

$$- \exp(\pm \lambda z) \sum_{n=0}^{\infty} b_n \frac{R^{n+1}}{2n-1} \frac{(\pm 1)^n}{n!} \Big\{ - \Big[\pm (2n-1)\lambda z - (n-1)^2 \Big] J_1(\lambda\rho)\, \mathbf{e}_\rho +$$

$$+ \Big[-(2n-1)\lambda z \pm n^2 \Big] J_0(\lambda\rho)\, \mathbf{e}_z \Big\}\Big\}\, d\lambda\,, \quad (z \neq 0)\,,$$

$$\mathbf{U}_1 = -\sigma_\alpha \sum_{k=0}^{\infty} \Big\{ b_k \frac{R^{k+1}}{2k+1} r^{-k} \Big[(1-k) P_k(\cos\theta)\, \mathbf{e}_r + P_k^{(1)}(\cos\theta)\, \mathbf{e}_\theta \Big] +$$

$$+ \frac{(-1)^k}{k!} r^{-k} \Big[\Big(kh\beta_{k-1} - k \frac{k^2-1}{2k-1} \beta_{k-2} - \frac{k+2}{2k+3} r^2 \beta_k \Big) P_k(\cos\theta)\, \mathbf{e}_r +$$

$$+ \Big(h\beta_{k-1} - \frac{k^2-1}{2k-1} \beta_{k-2} - \frac{r^2 \beta_k}{2k+3} \Big) P_k^{(1)}(\cos\theta)\, \mathbf{e}_\theta \Big]\Big\}\,,$$

$$\beta_k = \int_0^\infty B(\lambda) \exp(-\lambda h)\, \lambda^k\, d\lambda\,.$$

We pass over to the construction of the general solution of the homogeneous equation (2). We look for $\mathbf{U}_0$ in the form of the superposition of the outer (for a sphere) and inner (for a half-space) basis solutions of the Lamé equation:

$$\mathbf{U}_0 = \sum_{m=-1}^{1} \sum_{s=1}^{3} \Bigg[\int_0^\infty A_{m,s}(\lambda)\, \mathbf{U}_{s,\lambda,m}^{-(2)}\, d\lambda + \sum_{n=|m|}^{\infty} a_{n,m}^{(s)}\, \mathbf{U}_{s,n,m}^{+(4)} \Bigg],$$

$$\mathbf{U}_{s,n,m}^{\pm(2)} = \omega_s\, \mathbf{D}_s\, u_{\lambda,m}^{\pm(2)}\,, \quad s = 1, 2, 3\,, \quad \lambda \in R\,, \quad m \in Z\,;$$

$$\omega_1 = \omega_3 = \lambda^{-1}\,, \quad \omega_2 = 1\,; \quad \mathbf{D}_1 = \nabla\,, \quad \mathbf{D}_2 = z\nabla - \chi \mathbf{e}_z\,, \quad \mathbf{D}_3 = i\,[\nabla \times \mathbf{e}_z]\,;$$

$$\mathbf{U}_{1,n,m}^{+(4)} = \nabla u_{n,m}^{+(4)}\,, \quad \mathbf{U}_{2,n,m}^{+(4)} = [-(\chi(2n+1)+1)\, r\mathbf{e}_z + (n-\chi-1)\, r^2 \nabla]\, u_{n,m}^{+(4)}\,,$$

$$\mathbf{U}_{3,n,m}^{+(4)} = i\,[\nabla \times r u_{n,m}^{+(4)}\, \mathbf{e}_r]\,, \quad \mathbf{U}_{k,n,m}^{-(4)} = \mathbf{U}_{k,-n-1,m}^{+(4)}\,, \quad \chi = 3 - 4\sigma\,.$$

The following theorems are proved.

Theorem 1. When $n \geq |m|$, $z \neq 0$, the presentations of the outer spherical solutions of the Lamé equation in terms of the cylindrical solutions hold:

$$\mathbf{U}_{s,n,m}^{+(4)} = \frac{(-1)^m (\pm 1)^{n+m}}{(n-m)!} \sum_{j=0}^{3} \int_0^\infty W_{s,n,m}^{\pm(42)j,\lambda}\, \mathbf{U}_{j,\lambda,m}^{\mp(2)}\, d\lambda\,,$$

where

$$W_{1,n,m}^{\pm(42)j,\lambda} = \lambda^{n+1}\,\delta_{j1}\,;$$

$$W_{2,n,m}^{\pm(42)j,\lambda} = \lambda^{n-1}\left\{-\left[(1-\chi)\,(n^2-n+m^2)+n\,((n-1)^2-m^2)\right]\delta_{j1} \pm n\,(2n-1)\,\delta_{j2}+(1+\chi)\,m\,(2n-1)\,\delta_{j3}\right\};$$

$$W_{3,n,m}^{\pm(42)j,\lambda} = \pm\,\lambda^{n}\,(-m\delta_{j1}+n\delta_{j3})\,.$$

Theorem 2. The expansions of the solutions of the Lamé equation for a half-space with respect to the spherical basis solutions hold:

$$\mathbf{U}_{s,\lambda,m}^{-(2)} = \exp(-\lambda h)\sum_{j=1}^{3}\sum_{k=|m|}^{\infty}(-1)^k\frac{\lambda^{k-1}}{(k+m)\,!}\,W_{s,\lambda,m}^{-(24)j,\lambda}\,\mathbf{U}_{j,k,m}^{-(4)}\,,$$

where

$$W_{1,\lambda,m}^{-(24)j,k} = \delta_{j1}\,;$$

$$W_{2,\lambda,m}^{-(24)j,k} = \left[\lambda h-\frac{(k-\chi-1)\,(k^2-m^2)}{(2k-1)\,k}\right]\delta_{j1}-\frac{\lambda^2\delta_{2j}}{(2k+3)\,(k+1)}-\frac{n\,(\chi+1)}{k\,(k+1)}\,\lambda\delta_{j3}\,;$$

$$W_{3,\lambda,m}^{-(24)j,k} = -\left(\frac{m}{k}\,\delta_{j1}+\frac{\lambda}{k+1}\,\delta_{j2}\right).$$

The formulas given above make it possible to transform the displacement vector **U** separately to the spherical and cylindrical coordinates:

$$\mathbf{U} = \sum_{m=-1}^{1}\sum_{s=1}^{3}\int_0^{\infty}\left[A_{m,s}(\lambda)\,\mathbf{U}_{s,\lambda,m}^{-(2)}-\mathbf{U}_{s,\lambda,m}^{+(2)}\sum_{j=1}^{3}\sum_{n=|m|}^{\infty}a_{n,m}^{(s)}\,\frac{(-1)^n}{(n-m)\,!}\,W_{j,n,m}^{-(42)s,\lambda}\right]d\lambda\,-$$

$$-\,\sigma_\alpha\sum_{n=0}^{\infty}b_n\,\frac{R^{n+1}\,(-1)^{n+1}}{(2n-1)\,n\,!}\int_0^{\infty}\exp(-\lambda h)\,[(2n-1)\,\lambda h-n^2]\,\times$$

$$\times\,[J_1(\lambda\rho)\,\mathbf{e}_\rho-J_0(\lambda\rho)\,\mathbf{e}_z]\,\lambda^{n-1}\,d\lambda\,,$$

$$\mathbf{U} = \sum_{m=-1}^{1}\sum_{s=1}^{3}\sum_{n=|m|}^{\infty}\left[a_{n,m}^{(s)}\,\mathbf{U}_{s,n,m}^{+(4)}+\frac{(-1)^n}{(n+m)\,!}\sum_{j=1}^{3}\int_0^{\infty}A_{m,j}(\lambda)\exp(-\lambda h)\,\lambda^{n-1}\,W_{j,\lambda,m}^{-(24)s,n}\,d\lambda\,\mathbf{U}_{s,m,n}^{-(4)}\right]-$$

$$-\,\sigma_\alpha\sum_{k=0}^{\infty}\left\{\frac{b_kR^{k+1}}{(2k-1)\,r^k}\left[(1-k)\,P_k(\cos\theta)\,\mathbf{e}_r+P_k^{(1)}(\cos\theta)\,\mathbf{e}_\theta\right]+\frac{(-1)^k}{k!}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\,R^{n+1}\,r^{k-1}}{n!\,(2h)^{n+k-1}}\,(n+k-2)\,!\,\times\right.$$

$$\times\left[\left(\frac{k\,(n+k-1)}{2}-k\,\frac{(k-1)^2}{2k-1}-\frac{r}{h}\,\frac{2k+2}{2k+3}\,\frac{(n+k)\,(n+k-1)}{4}\right)P_k(\cos\theta)\,\mathbf{e}_r\,+\right.$$

$$\left.\left.+\left(\frac{(n+k-1)}{2}-\frac{(k-1)^2}{2k-1}-\left(\frac{r}{h}\right)^2\frac{(n+k)\,(n+k-1)}{4\,(2k+3)}\right)P_k^{(1)}(\cos\theta)\,\mathbf{e}_\theta\right]\right\}.$$

TABLE 1. Values of the Coefficients R_1, R_2, and R_3 at Different R/h

R/h	0.1	0.3	0.5	0.7	0.8
R_1	2.902	3.252	4.169	6.697	9.668
R_2	12.70	9.385	7.031	5.800	5.539
R_3	1.905	1.907	1.948	2.287	2.837

Satisfaction of conditions (2)–(4) leads to an infinite system of linear algebraic equations relative to unknown coefficients $a_{n,m}^{(s)}$:

$$\sum_{s=1}^{2}\left\{a_{k,m}^{(s)}q_{k,p}^{(s)}+r_{k,p}^{(s)}\left[\nu_{k,p}^{(s)}+\sum_{j=1}^{3}\sum_{n=\mu}^{\infty}\tau_{k,n,m}^{s,j}\,a_{n,m}^{(j)}\right]\right\}=\delta_{m0}\sigma_{\alpha}d_{k,p}\,,$$

$$a_{k,m}^{(3)}+\sum_{j=1}^{3}\sum_{n=|m|}^{\infty}\tau_{k,n,m}^{(3,j)}\,a_{n,m}^{(j)}=\nu_{k,m}^{(3)}\,,\quad k\geq |m|\,,\quad p=1,2\,. \tag{6}$$

The matrix coefficients of the system can be found rather easily, but their explicit form allows one to show that under the conditions of $R<h$ the operator of system (6) is the Fredholm operator, which makes it possible to solve the system numerically by the reduction method.

Using the above-described method we can find the elastic stressed-deformed state with an arbitrary axisymmetric temperature field.

The force P and the momentum which are applied to the die are related to the displacement δ and the angle of rotation of the die γ (3) by the following formulas:

$$\frac{P}{2G}=\delta aR_1+\alpha Ta^2R_2\,,\quad \frac{\rho_0 P}{2G}=\gamma a^3R_3\,.$$

Table 1 gives the values of the parameters R_1, R_2, and R_3 obtained at a/R = 0.5.

The stresses in the region under consideration can be presented in the form of the superposition of the stresses due to axisymmetric indentation of the die into the half-space σ^0, rotation of the die σ^1, and the temperature expansion σ^T. For example, under the die we have

$$\frac{a^2}{P}\sigma_z(\rho,\varphi)=\sigma_z^0+\frac{\rho_0}{a}\sigma_z^1\cos\varphi+\frac{2Ga^2}{P}\alpha_T T_0\sigma_z^T\,.$$

The character of the stressed-deformed state under the die can be judged by the graphs (the stresses were found at a/R = 0.5) given in Fig. 1.

A numerical analysis of the problem shows the presence of zones of tensile thermoelastic stresses in the region of contact between the die and the half-space and on the core when they all are rather close to each other ($R/h\geq 0.3$). The latter is a consequence of the fact that, under the effect of the linear temperature expansion caused by heating of the regions between the core and the boundary of the half-space, the boundary of the half-space moves aside from the core. In this case, the value of the total stress $\sigma_z(\rho, 0, \varphi)$ on the die axis decreases in magnitude. This can occur only because of the fact that the temperature stresses σ^T near the die axis are positive.

It is of interest that the radius of the zone of tensile stresses decreases with an increase in the ratio a/R. The highest concentration of stresses in the vicinity of the die axis is observed at a small relative size of the core.

Independently, we obtained an approximate solution of the system in the form of a series with respect to the small parameter ε = R/h. We give some results:

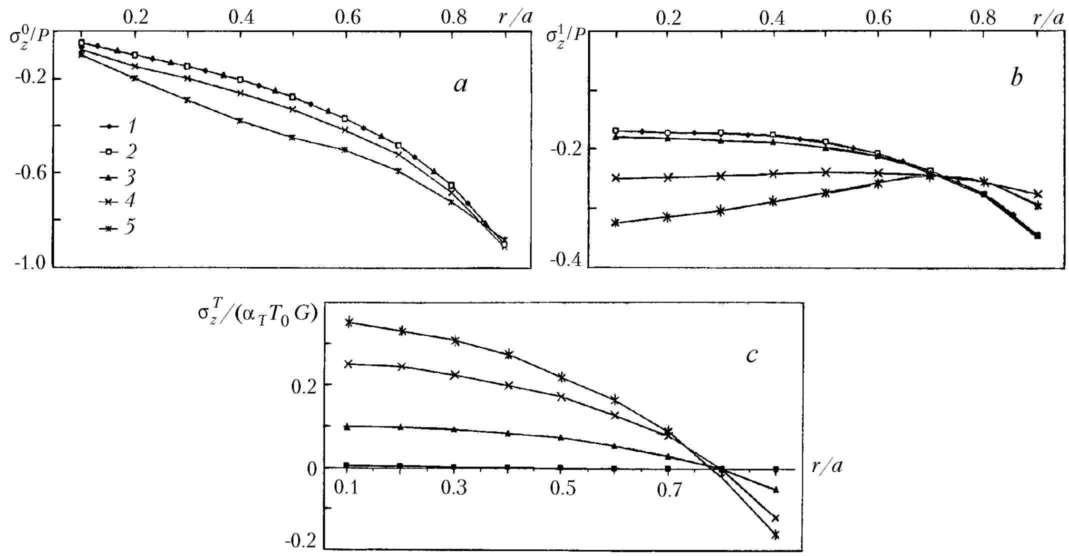

Fig. 1. Stresses σ_z^0/P (a), σ_z^1/P (b), and $\sigma_z^T/(\alpha_T T_0 G)$ (c) under the die: 1) R/h = 0.1; 2) 0.3; 3) 0.5; 4) 0.7; 5) 0.8.

$$R_1 = \frac{2}{1-\sigma}\left\{1 + \frac{12\varepsilon}{(10-12\sigma)\,\pi}\left[(2-2\sigma)\arctan\eta + \frac{\eta}{1+\eta^2}\right]^2\right\} + O(\varepsilon^2)\,,$$

$$R_2 = 4\cdot\frac{1+\sigma}{1-\sigma}\,\eta^{-2}\,\varepsilon\arctan\eta + O(\varepsilon^2)\,,$$

$$R_3 = \frac{4}{3\,(1-\sigma)}\left\{1 + \frac{18\varepsilon\eta}{(10-12\sigma)\,\pi}\left[(2-2\sigma)\,\frac{\eta-\arctan\eta}{\eta^2}\ \frac{\eta}{1+\eta^2}\right]^2\right\} + O(\varepsilon^2)\,,$$

$$\sigma_z^0(0) = -\,(\pi\,(1-\sigma)\,R_1)^{-1}\left\{1 + \frac{12\varepsilon}{(10-12\sigma)\,\eta}\left[(2-2\sigma)\arctan\eta + \frac{\eta}{1+\eta^2}\right]\times\right.$$

$$\left.\times\left[(2-2\sigma)\,(\eta\arctan\eta + 1) + 2\eta\arctan\eta + \frac{\eta}{1+\eta^2}\right]\right\} + O(\varepsilon^2)\,,$$

$$\sigma_z^T(0) = \frac{2\,(1+\sigma)}{\pi\,(1-\sigma)}\,\eta^{-2}\,\varepsilon\,[(\eta^2-1)\arctan\eta + \eta] + O(\varepsilon^2)\,,\quad \eta = a/h\,.$$

We note that the stresses found by these formulas are in agreement with the numerical results at $\varepsilon = 0.1$ obtained earlier.

NOTATION

h, distance from the boundary of the half-space to the center of the spherical inclusion; *R*, radius of the inclusion; $\varepsilon = R/h$; (r, θ, φ) and (ρ, z, φ), spherical and cylindrical systems of coordinates related to the inclusion and the half-space, respectively; (r_1, θ_1, φ), (ρ_1, z_1, φ), co-directional systems of coordinates with a common origin; *a*, radius of the in-plan round die; $\eta = a/h$; *P*, force applied to the die at the point $(\rho_0, 0, 0)$; σ, Poisson coefficient; α_T, coefficient of linear temperature expansion; *G*, shear modulus; *T*, temperature field; T_0, temperature of the inclusion; σ_z, $\tau_{\rho z}$, $\tau_{\varphi z}$, components of the stress tensor; U_z, component of the displacement vector; δ and γ, die displacement along the axis 0z and the angle of rotation of the die about the axis 0y; $\Omega = \{z > 0, r > R\}$, region of the stressed-deformed state under investigation; $B(\lambda)$ and b_n, unknown weighting function and the coefficients in the heat-conduction problem; $u^{\pm(2)}_{\lambda,m}$ and $u^{\pm(4)}_{n,m}$, outer (+) and inner (–) basis harmonic functions for the half-space and the sphere, respectively; $P^m_n(x)$, joint Legendre function of first kind; J_m, Bessel function of first kind; n, m, integral indices which are discrete spectral parameters of the solution of the Laplace equation, $\mu = |m|$; λ, continuous spectral parameter ($\lambda \in R$) of the solution of the Laplace equation for the half-space; k, nonnegative integral index; δ_{kn}, Kronecker symbol; ($\mathbf{e}_\rho$, $\mathbf{e}_z$) and ($\mathbf{e}_\theta$, $\mathbf{e}_r$), unit vectors of the cylindrical and spherical systems of coordinates; $\mathbf{U}$, vector of thermoelastic displacements; $\mathbf{U}_1$, partial solution of the inhomogeneous Lamé equation; $\mathbf{U}_0$, general solution of the homogeneous Lamé equation; Φ, thermoelastic potential; $\mathbf{U}^{\pm(42)j,\lambda}_{s,n,m}$ and $\mathbf{U}^{\pm(2)}_{s,\lambda,m}$, basis solutions of the homogeneous Lamé equation for the sphere and the half-space, respectively; s, number of linearly independent basis solutions of the homogeneous Lamé equation; i, imaginary unit; $A_{m,s}(\lambda)$ and $a^{(s)}_{n,m}$, unknown weighting functions and coefficients in the basis solutions of the Lamé equation; R_s, parameters in the formulas relating force P and distance ρ_0 to displacements δ and γ; $O(\varepsilon^2)$, infinitesimal quantity of the order of smallness higher than ε^2; $W^{\pm(42)j,\lambda}_{s,n,m}$ and $W^{-(24)j,\lambda}_{s,\lambda,m}$, coefficients of the vector summation theorems relating the basis solutions of the Lamé equations for the sphere and the half-space.

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